Isomorphism Problems and Minimum Circuit Size

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Outline

Characters:

(1) Graph Isomorphism and generalizations

(2) Minimum Circuit Size and related problems

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Result: Reductions from (1) to (2)

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Instantiations:

Isomorphism Problem	H	Ω
Discrete Log	$\mathbb{F}_p$	$\mathbb{F}_p$
Graph Isomorphism	S_n	graphs with n vertices
Linear Code Equivalence	S_n	subspaces of $\mathbb{F}_q^n$
Permutation Group Conjugacy	S_n	subgroups of S_n
Matrix Subspace Conjugacy	$\mathrm{GL}_n(\mathbb{F}_q)$	subspaces of $\mathbb{F}_q^{n \times n}$

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"Hard" problems underlying PRG constructions that fool circuits/efficient programs reduce to $\mathrm{MCSP}/\mathrm{MKTP}$.

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$$\Pr[C(\tau) = C(\sigma)] \geq 1/\mathrm{poly}(n) \text{ with } \tau \doteq M(C, C(\sigma)).$$

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If $G_0 \equiv G_1$ then $\sigma(G_1) \equiv \sigma(G_0)$, so

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$$\Pr[C(\tau) = \sigma(G_0)] \geq 1/\mathrm{poly}(n) \text{ with } \tau \doteq M(C, \sigma(G_0)).$$

$C(\tau) = \sigma(G_0)$ means $\tau^{-1} \circ \sigma$ is an isomorphism from G_0 to G_1.

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Approach: Use $\mathrm{MCSP}/\mathrm{MKTP}$ to *verify* witnesses of nonisomorphism.

Witnessing Rigid Graph Nonisomorphism

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Consider sampling $r \sim \{0,1\}$ and $\pi \sim S_n$ uniformly, and outputting the adjacency matrix of $\pi(G_r)$.

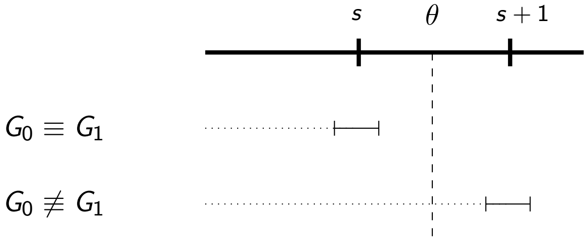
- If $G_0 \equiv G_1$, this has entropy $s \doteq \log n!$
- If $G_0 \not\equiv G_1$, this has entropy $s + 1$

Main idea: Use KT-complexity of a random sample as a proxy for entropy.

Witnessing Rigid Graph Nonisomorphism

Let $y = \pi(G_r)$, $\pi \sim S_n$, $r \sim \{0, 1\}$.

Hope: $\mathrm{KT}(y)$ is typically near the entropy, never much larger:



where $s = \log(n!)$

Witness nonisomorphism by checking $\mathrm{KT}(y) > \theta$.

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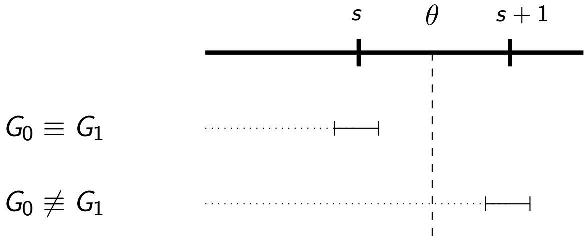
Let $y = \pi_1(G_{r_1})\pi_2(G_{r_2})\cdots\pi_t(G_{r_t})$, $\pi_i \sim S_n$, $r_i \sim \{0,1\}$.

Reality: $\mathrm{KT}(y)/y$ is typically near the entropy, never much larger:



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Implies that $\mathrm{KT}(y)/t \leq s + o(1)$ for sufficiently large $t = \mathrm{poly}(n)$.

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As y consists of t independent samples from a flat distribution of entropy s, $\mathrm{KT}(y)/t \geq s - o(1)$ holds with high probability.

Corollary: $KT(y)/t$ is a *probably approximately correct (PAC) underestimator* for the entropy.

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Let $y = \pi_1(G_{r_1})\pi_2(G_{r_2})\cdots\pi_t(G_{r_t})$, $\pi_i \sim S_n$, $r_i \sim \{0,1\}$.

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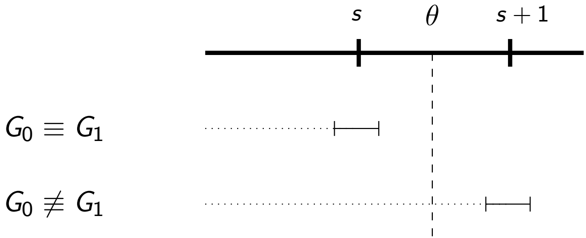

where $s = \log(n!/|\mathrm{Aut}(G_i)|)$ (assumed the same for $i \in \{0,1\}$ for ease of exposition).

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Using entropy estimator yields randomized reduction with two-sided error (from which false positives can be eliminated using known search-to-decision for Graph Isomorphism).

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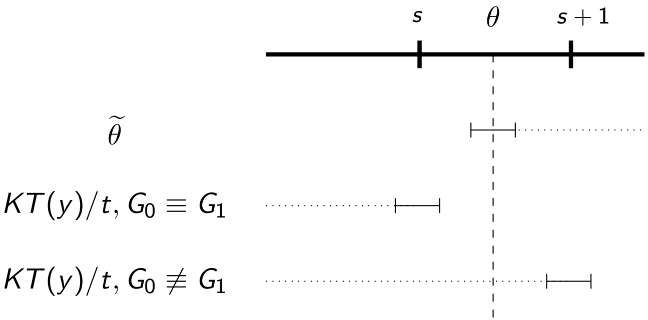
It suffices to give a probably approximately correct (PAC) *over*estimator for θ:



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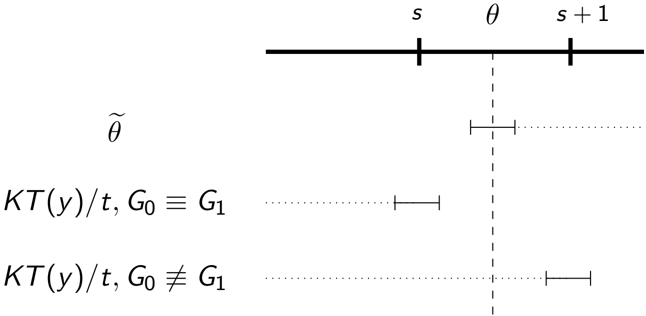

Equivalently, it suffices to give a PAC *under*estimator for $\log|\mathrm{Aut}(G_i)|$, since $\theta = \log(n!) - \log|\mathrm{Aut}(G_i)|$.

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$|A|$ can be computed efficiently from its generators. Output $\log|A|$.

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Theorem: Graph Isomorphism is in $\mathrm{ZPP}^{\mathrm{MKTP}}$.

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Need:

- Encoding of the cosets of $\mathrm{Aut}(\omega)$ that achieves KT-complexity close to the entropy s.
- Efficiently compute $|H|$.
- PAC underestimator for $|\mathrm{Aut}(\omega)|$ that is efficiently computable with access to MKTP.

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Encoding Lemma: Let C be a circuit sampling a distribution of max-entropy s. There is a circuit D of size poly($|C|$) and, for each outcome z, a string i_z of length $s + \log s + O(1)$ s. t. $D(i_z) = z$.

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Corollary: If y is the concatenation of t independent samples from a samplable distribution p then for any sufficiently large polynomial t, $\mathrm{KT}(y)/t$ is a PAC underestimator for the entropy of p with deviation $\Delta + o(1)$ where $\Delta =$ max-entropy - min-entropy of p.

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Approach: Efficient procedures with access to MKTP to compute:

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Consider sampling h' uniformly from H, and running $M(C, h'(\omega))$

On success, we get h so that $h(\omega) = h'(\omega)$, *i.e.*, $h^{-1}h' \in \mathsf{Aut}(\omega)$

In fact, conditioned on M's success, $h^{-1}h'$ is *uniform* on $\mathsf{Aut}(\omega)$

Only polynomially-many trials are needed to obtain a full set of generators with high probability.

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A sequence $h_1, h_2, \ldots, h_k$ of elements of a finite group Γ is said to be *Erdős–Rényi* if the random subproduct

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[Erdős Rényi] showed that every finite group has such a generating set of size $\mathrm{poly}(\log |\Gamma|)$, and [Babai] gave an efficient randomize algorithm to find one with overwhelming probability. This gives the required circuit.

Generic Result

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Theorem. Let Iso be an Isomorphism Problem satisfying the following conditions:

- The action $(h, \omega) \mapsto h(\omega)$, products and inverses in H, and $|H|$ can be computed efficiently.
- The uniform distribution on H can be sampled efficiently.
- There is an efficiently computable complete invariant for H.
- There is an efficiently computable complete invariant for Ω.

Then $\mathrm{Iso} \in \mathsf{ZPP}^{\mathrm{MKTP}}$.

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Thanks!