On the PPA-completeness of the Combinatorial Nullstellensatz and the Chevalley-Warning Theorem

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joint work with

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Overview of the talk

1. The class PPA
2. CNSS and Chevalley-Warning Theorem
3. Arithmetic circuits and parse subcircuits
4. The problems PPA-Circuit Chevalley and PPA-Circuit CNSS
5. PPA-hardness and PPA-easiness

The class PPA

Functional NP (FNP)

NP-search problems are defined by binary relations

$$R \subseteq \{0,1\}^* \times \{0,1\}^* \text{ such that}$$

– $R \in P$,
– for some polynomial $p(n)$, $R(x,y) \implies |y| \le p(|x|)$.

SEARCH PROBLEM Π_R
Input: x
Output: A solution y such that $R(x,y)$ if there is any, or "failure"

Π_R is reducible to Π_S if there exist polynomial time computable functions f and g such that, for every positive x,

$$S(f(x),y) \implies R(x,g(x,y)).$$

Total Functional NP (TFNP) [MP'91]

An NP-search problem is total if for all x there exists a solution y.

Facts:

- If $\mathrm{FNP} \subseteq \mathrm{TFNP}$ then $\mathrm{NP} = \mathrm{coNP}$.
- If $\mathrm{TFNP} \subseteq \mathrm{P}$ then $\mathrm{P} = \mathrm{NP} \cap \mathrm{coNP}$.

TFNP is a semantic complexity class

Syntactical subclasses of TFNP:

- Polynomial Local Search PLS
  Examples: Local optima, pure equilibrium in potential games
- Polynomial Pigeonhole Principle PPP
  Examples: Pigeonhole SubsetSum, Discrete Logarithm
- Polynomial Parity Argument classes $\mathrm{PPA}, \mathrm{PPAD}$.

Polynomial Parity Argument [P'94]

Parity Principle: In a graph the number of odd vertices is even.

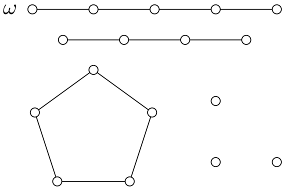
Definition: PPA is the set of total problems reducible to LEAF

LEAF

Input: (z, M, ω), where

- z is a binary string
- M is a polynomial TM that defines a graph $G_z = (V_z, E_z)$
- $V_z = \{0,1\}^{p(|z|)}$ for some polynomial p
- for $v \in V_z$, $M(z, v)$ is a set of at most two vertices
- $\{v, v'\} \in E_z$ if $v' \in M(z, v)$ and $v \in M(z, v')$
- $\omega \in V_z$ is a degree one vertex, the standard leaf

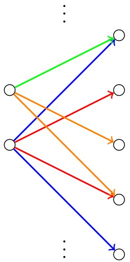
Output: A leaf different from ω.



PPA with edge recognition and pairing

Graphs $G_z = (V_z, E_z)$ of unbounded degree can be defined by two polynomial time algorithms ϵ and ϕ:

- Edge recognition: $\{v, v'\} \in E_z \Leftrightarrow \epsilon(v, v') = 1$
- Pairing: For every vertex v,
  - if $\deg(v)$ is even the function $\phi(v, \cdot)$ is a pairing between the vertices adjacent to v.
  - if $\deg(v)$ is odd then
    there exists exactly one neighbor w such that $\phi(v, w) = w$, and on the remaining neighbors $\phi(v, \cdot)$ is a pairing.



Fact: A problem defined in terms of ϵ and π is in PPA.

Proof: Let $G'_z = (V'_z, E'_z)$ be defined as

- $V'_z = E_z$
- $\{\{v, w\}, \{v, w'\}\} \in E'_z$
  if $\phi(w) = w'$.

Examples of problems in PPA

Few complete problems are known, mostly discretizations or combinatorial analogues of topological fixed point theorems:

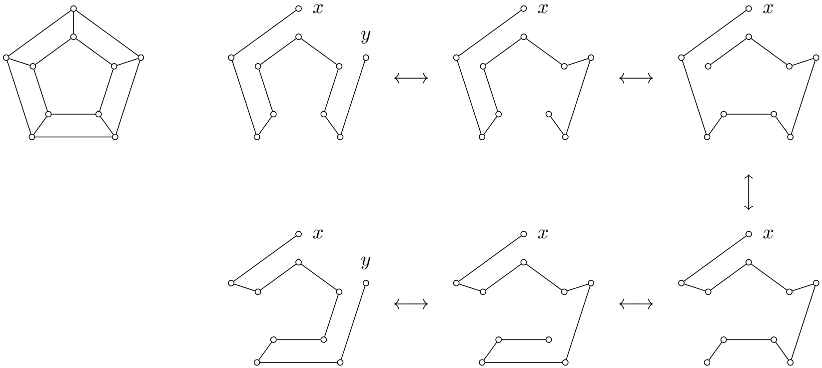
- 3-D Sperner in some non-orientable space [G'01]
- Locally 2-D Sperner [FISV'06]
- 2-D Tucker in the Euclidean space [ABB'15]
- Sperner and Tucker on the Möbius band [DEFLQX16]
- Octrahedral Tucker [DFK17]
- Consensus Halving [F-RG17]

Many problems of various origins are in PPA:

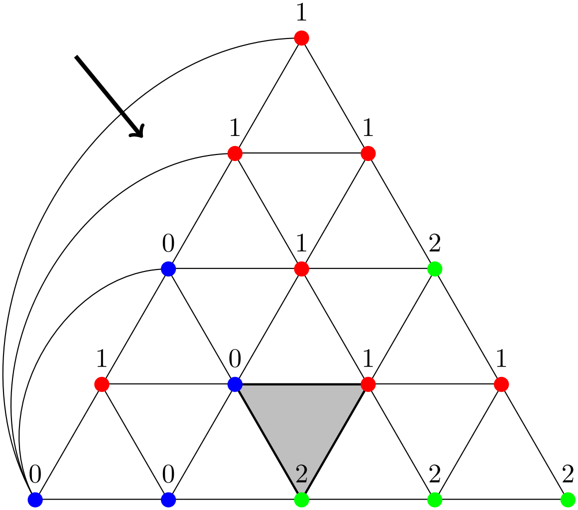
- Graph theory: Smith, Hamiltonian decomp. [P'94]
- Combinatorics: Necklace splitting and Discrete ham sandwich [P'94]
- Algebra: Explicit Chevalley [P'94]
- Number theory: Square root and Factoring [J'16]

Example: Smith

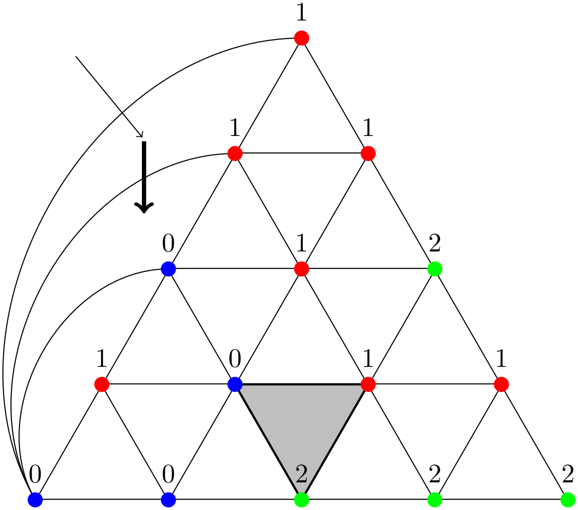
Theorem: In a cubic graph, for every edge, there is an even number of Hamiltonian cycles going trough the edge



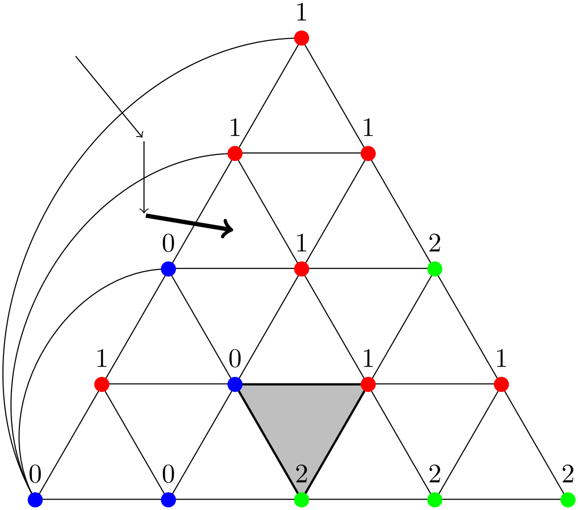
Example: SPERNER LEMMA



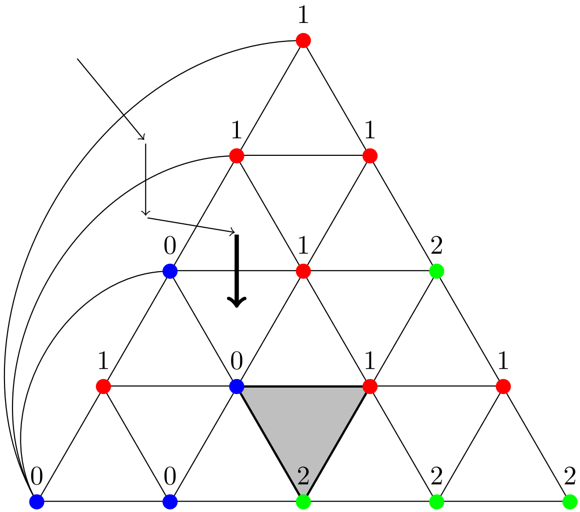
Example: Sperner Lemma



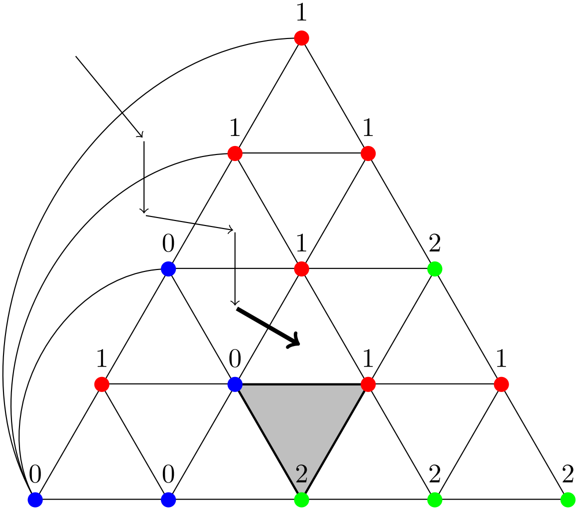
Example: Sperner Lemma



Example: Sperner Lemma



Example: Sperner Lemma



Combinatorial Nullstellensatz and Chevalley-Warning Theorem

Combinatorial Nullstellensatz

Theorem [Alon'99]: Let $\mathbb{F}$ be a field, let $d_1, \ldots, d_n$ be non-negative integers, and let $P \in \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial. Suppose that

- $\deg(P) = \sum_{i=1}^{n} d_i$,
- the coefficient of $x_1^{d_1} \ldots x_n^{d_n}$ is non-zero.

Then for every subsets $S_1, \ldots, S_n$ of $\mathbb{F}$ with $|S_i| > d_i$, there exists $(s_1, \ldots s_n) \in S_1 \times \cdots \times S_n$ such that

$$P(s_1, \ldots, s_n) \neq 0.$$

Consequences in algebra, graph theory, combinatorics, additive number theory ...

Chevalley-Warning Theorem

Theorem [Chevalley'36, Warning'36]: Let $\mathbb{F}$ be a field of characteristic p, and let $P_1, \ldots, P_k \in \mathbb{F}[x_1, \ldots, x_n]$ be non-zero polynomials.

If $\sum_{i=1}^{k} \deg(P_i) < n$, then the number of common zeros of $P_1, \ldots, P_k$ is divisible by p.

In particular, if the polynomials have a common root, they also have another one.

The theorems over $\mathbb{F}_2$

Definition A multilinear polynomial over $\mathbb{F}_2$ is

$$M(x_1, \ldots, x_n) = \sum_{T \subseteq \{1,\ldots,n\}} c_T \prod_{i \in T} x_i, \quad \text{where} \quad c_T \in \mathbb{F}_2$$

Fact: For every P over $\mathbb{F}_2$, there exists a unique multilinear polynomial M_P such that P and M_P compute the same function.

Definition: The multilinear degree of P is $\mathrm{mdeg}(P) = \deg(M_P)$.

Theorem [Combinatorial Nullstellensatz over $\mathbb{F}_2$]: Let P be such that $\mathrm{mdeg}(P) = n$.
Then there exists $a \in \mathbb{F}_2^n$ such that $P(a) = 1$.

Theorem[Chevalley-Warning over $\mathbb{F}_2$]: Let P such that $\mathrm{mdeg}(P) < n$, and let $a \in \mathbb{F}_2^n$ such that $P(a) = 0$.
Then there exists $b \neq a$ such that $P(b) = 0$.

Theorem: $\mathrm{mdeg}(P) < n \iff$ the number of zeros is even

How to make them search problems?

Theorem[P'94]: The following problem is in PPA.

EXPLICIT CHEVALLEY

Input: Explicitly given polynomials $P_1, \ldots, P_k$ over $\mathbb{F}_2$ such that

$$\sum_{i=1}^{k} \deg(P_i) < n,$$

and a common root $a \in \mathbb{F}_2^n$.

Output: Another common root $a' \neq a$.

Remark: a is common root $\Leftrightarrow P(a) = 0$ where

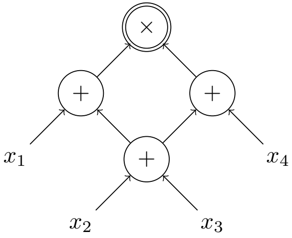
$$P = 1 + \prod_{i=1}^{k} (P_i + 1)$$

Could this be PPA-hard? Probably not. Two restrictions:

- P is given by an arithmetic circuit of specific form
- even the degree of P is less than n

Arithmetic circuits and parse subcircuits

Arithmetic circuits



C is a labeled, directed, acyclic graph.

Labels $= \{+, \times\}$,

$G^+ =$ sum gates, $G^\times =$ product gates.

Computational gates have indegree 2: left and right child

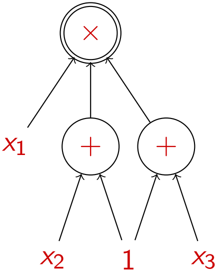
Polynomial computed by C

$$C(x) = (x_1 + x_2 + x_3) \times (x_2 + x_3 + x_4)$$
$$= x_1x_2 + x_1x_3 + x_1x_4 + x_2^2 + x_2x_4 + x_3^2 + x_3x_4$$

Lagrange-circuits

Circuits computing the Lagrange basis polynomials $L_a(x)$

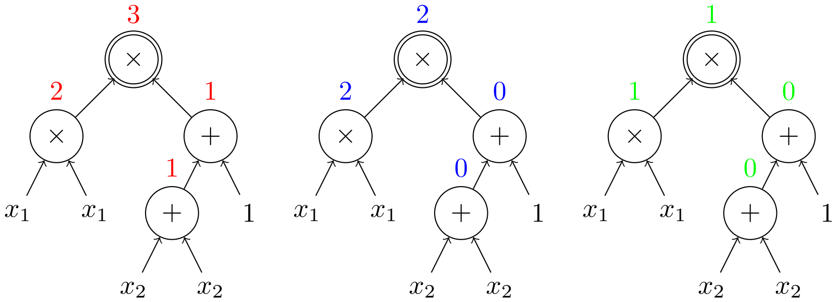
$$L_a(x) = 1 \iff x = a$$



Lagrange-circuit L_{100}

Degrees in a circuit

There are 3 types of degree



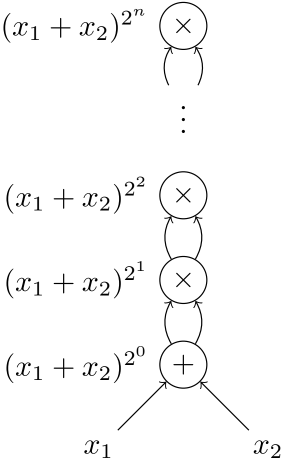
Formal degree = 3 Polynomial degree = 2 Multilinear degree =1

$2 = 0$ $x^2 = x$

easy to compute ??

We are interested in the multilinear degree

Multilinear degree and monomials



How can we certify $\mathrm{mdeg}(C(x)) = n$?

What is the the complexity of $\textsc{Mdeg} = \{C : \mathrm{mdeg}(C(x)) = n\}$?

We wish $\textsc{Mdeg} \in \mathrm{NP}$

A monomial m computed by C is maximal if $\mathrm{mdeg}(m) = n$

Fact: $\mathrm{mdeg}(C(x)) = n \iff$ odd number of maximal monomials

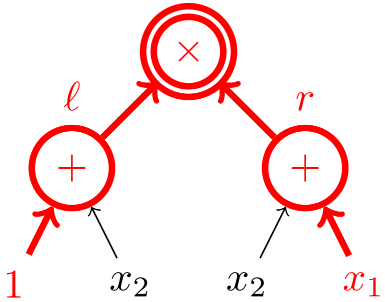
Difficulty: the number of monomials computed by C can be doubly exponential in the size of C

We can certainly say that $\textsc{Mdeg} \in \oplus\mathrm{EXP}$

Monomials in arithmetic formulae

Let F be an arithmetic formula

Monomials are computed by parse subtrees defined by the marking of appropriate sum gates: $S : G^{+} \to \{\ell, r, *\}$:



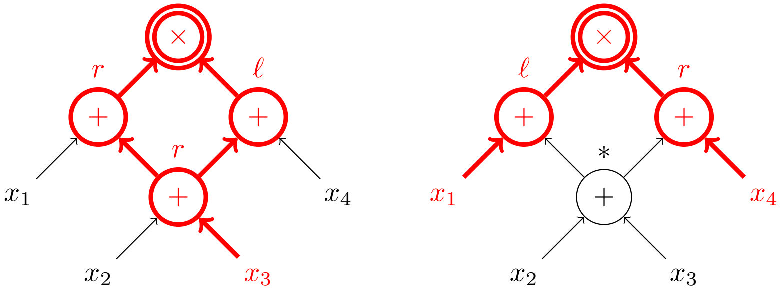
Parse subcircuits

C arithmetic circuit. A parse subcircuit is a partial marking

$$S : G^+ \to \{\ell, r, *\}$$

such that

marked vertices = accessible vertices



computes x_3^2 computes $x_1 x_4$

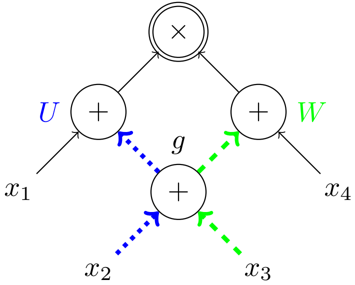
Parse subcircuits witness monomials

$\mathcal{S}(C) =$ set of parse subcircuits of C,

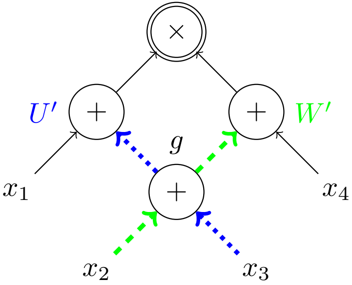
$m_S(x) =$ monomial computed by parse sub circuit S

Theorem: Let $\mathbb{F}$ be a field of characteristic 2. Then

$$C(x) = \sum_{S \in \mathcal{S}(C)} m_S(x).$$



$m_U m_W = x_2 x_3$



$m_{U'} m_{W'} = x_2 x_3$

Corollary: $\textsc{MDeg} \in \oplus\mathrm{P}$ Proposition: $\textsc{MDeg}$ is $\oplus\mathrm{P}$-hard.

The problems

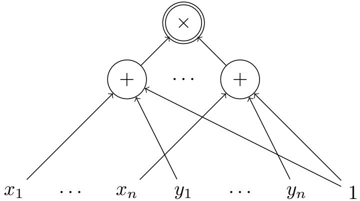
PPA-Circuit Chevalley

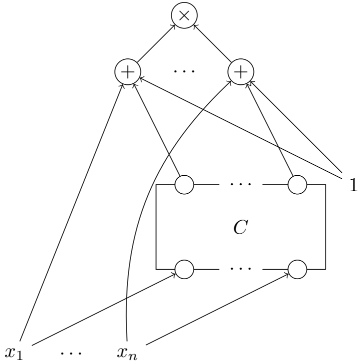
and PPA-Circuit CNSS

Towards PPA-circuits

We would like to characterize PPA with arithmetic circuits

Auxiliary circuits I and $I \diamond C$:





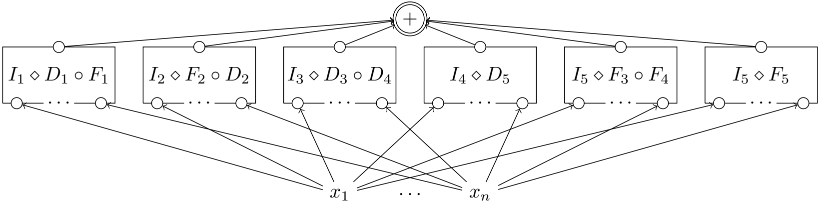
$$I(x_1, \ldots, x_n, y_1, \ldots, y_n) = \prod_{i=1}^{n}(x_i + y_i + 1)$$

$$I(x, y) = 1 \iff x = y$$

$$I \diamond C(x) = 1 \iff C(x) = x$$

PPA-circuits

Definition: A PPA-circuit is the PPA-composition $C_{D,F}$ of two n-variable, n-output arithmetic circuits D and F over $\mathbb{F}_2$



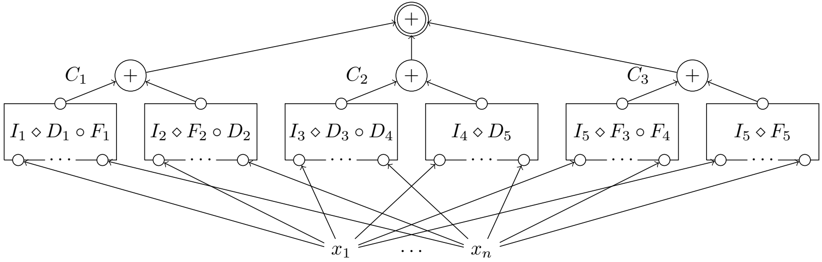
$C_{D,F}$

PPA-Circuit Matching Lemma:

If C is a PPA-circuit then in polynomial time a perfect matching μ can be computed between the maximal parse subcircuits of C.

PPA-Circuit Matching Lemma

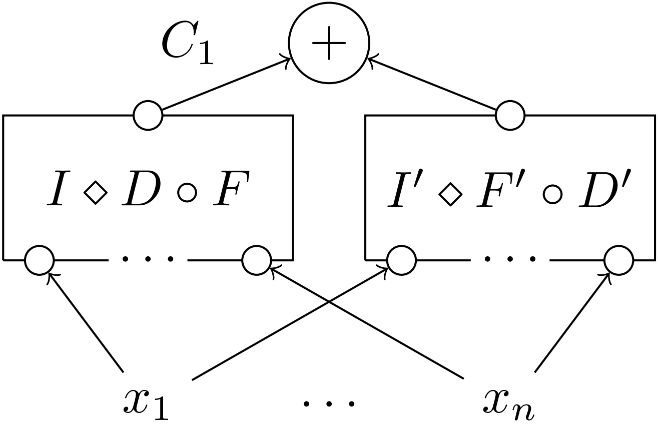
We want to define a polynomial time computable μ:
perfect matching on the maximal parse subcircuits of $C_{D,F}$



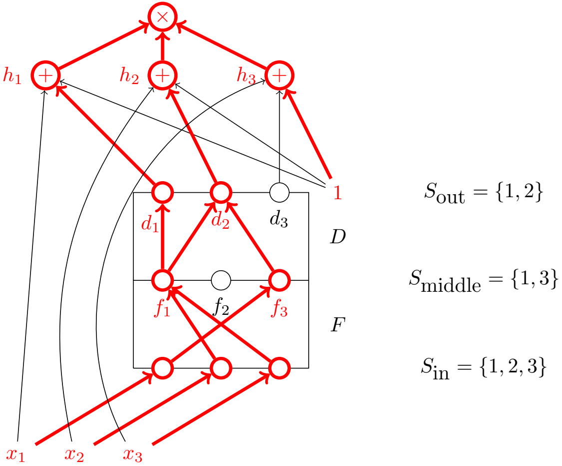
$$C_{D,F} = C_1 + C_2 + C_3$$

μ is defined inside C_1, inside C_2 and inside C_3

The matching μ inside C_1



The matching μ inside C_1



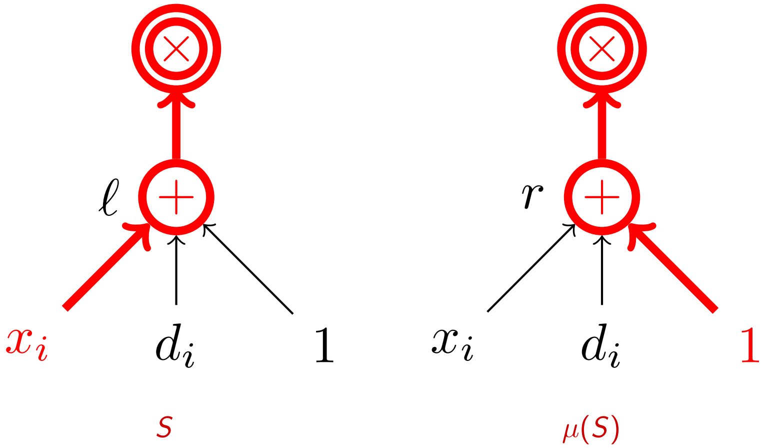
$i \in S_{\text{out}}$ if the edge from the d_i to h_i belongs to S
$i \in S_{\text{middle}}$ if there exists an edge in S from f_i to a gate in D
$i \in S_{\text{in}}$ if there exists an edge in S from x_i to a gate in F

Claim: $S_{\text{out}} \subseteq S_{\text{in}}$

The matching μ inside C_1

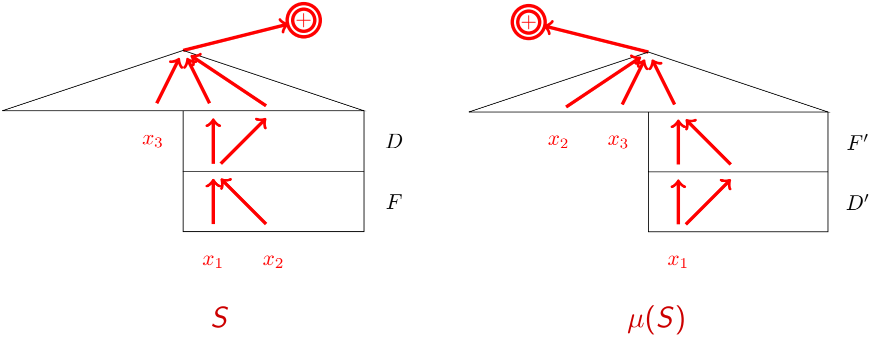
Case 1: $S_{\text{out}} \subset S_{\text{in}}$

Let i be the smallest index in $S_{\text{in}} \setminus S_{\text{out}}$

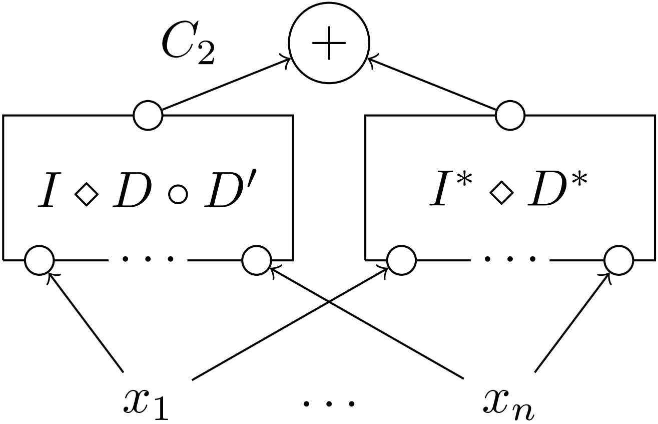


The matching μ inside C_1

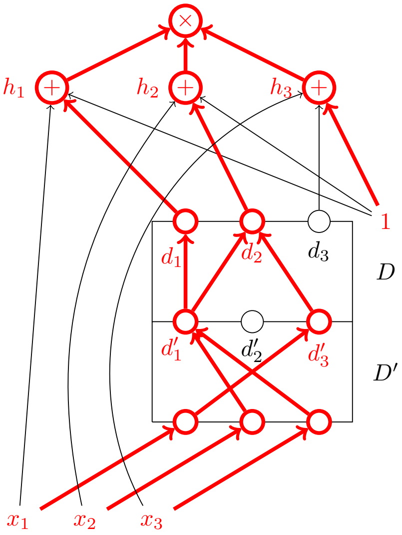
Case 2: $S_{\text{out}} = S_{\text{in}}$



The matching μ inside C_2



The matching μ inside C_2



$S_{\text{out}} = \{1, 2\}$

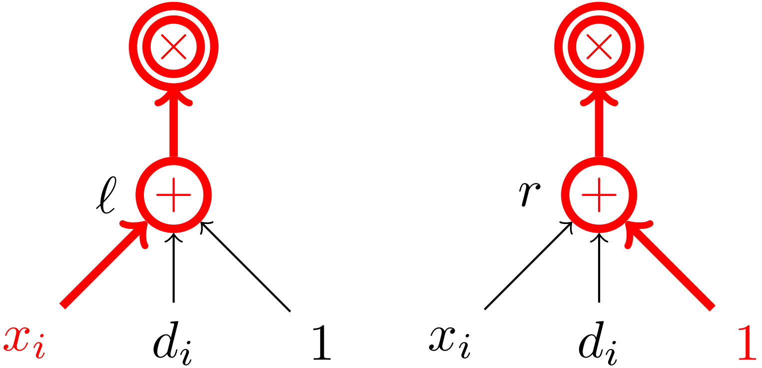
$S_{\text{middle}} = \{1, 3\}$

$S_{\text{in}} = \{1, 2, 3\}$

The matching μ inside C_2

Case 1: $S_{\text{out}} \subset S_{\text{in}}$

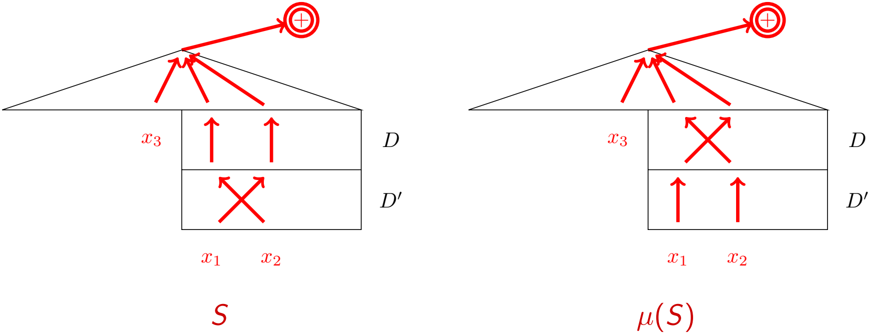
Let i be the smallest index in $S_{\text{in}} \setminus S_{\text{out}}$



S $\mu(S)$

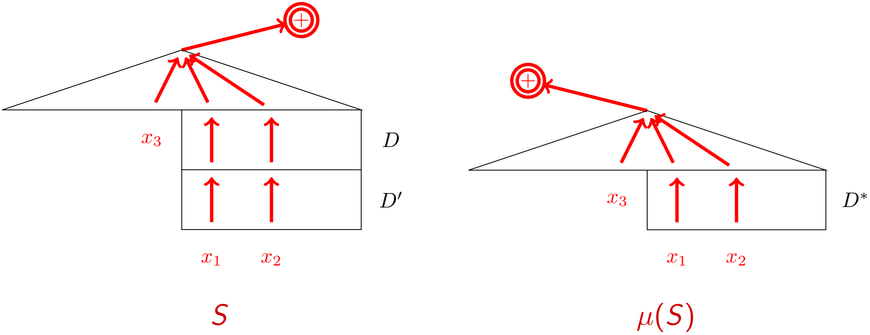
The matching μ inside C_2

Case 2: $S_{\text{out}} = S_{\text{in}}$ and $S(g) \neq S(g')$ for some sum gate in D



The matching μ inside C_2

Case 3: $S_{\text{out}} = S_{\text{in}}$ and $S(g) = S(g')$ for all sum gate in D



The computational problems

PPA-CIRCUIT CHEVALLEY

Input: (C, a), where
C: an n-variable PPA-circuit over $\mathbb{F}_2$,
a: a root of C.

Output: Another root $b \neq a$ of C.

PPA-CIRCUIT CNSS

Input: (C', a), where
C: an n-variable PPA-circuit over $\mathbb{F}_2$,
a: an element of $\mathbb{F}_2^n$.

Output: An element $b \in \mathbb{F}_2^n$ satisfying $C = C' \oplus L_a$.

The result

Main Theorem: PPA-Circuit Chevalley
and PPA-Circuit-CNSS are PPA-complete.

The proof contains three parts:

Proposition: PPA-Circuit Chevalley
and PPA-Circuit CNSS are polynomially equivalent.

Hardness Theorem: PPA-Circuit Chevalley is PPA-hard.

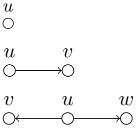
Easiness Theorem: PPA-Circuit CNSS is in PPA.

PPA-hardness and PPA-easiness

PPA-hardness

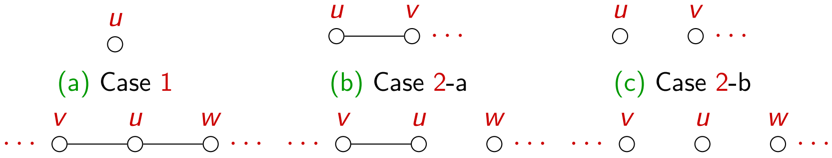
Theorem: PPA-Circuit Chevalley is PPA-hard.

Proof: Reduce Leaf to PPA-Circuit Chevalley
Express the ≤ 2 neighbours $M(u)$ of u via $D(u)$ and $F(u)$:



- Case 1: then $D(u) = F(u) = u$,
- Case 2: then $D(u) = v$ and $F(u) = u$,
- Case 3: then $D(u) = v$ and $F(u) = w$

Claim: Parity of $\deg(u)$ = Parity of satisfied components of $C_{D,F}$



(a) Case 1 (b) Case 2-a (c) Case 2-b

(d) Case 3-a (e) Case 3-b (f) Case 3-c

PPA-easiness

We prove something stronger

MATCHED-CIRCUIT CNSS

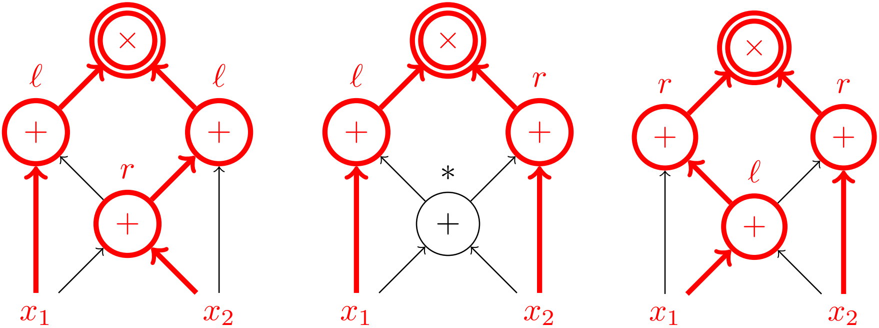
Input: (C, T, μ), where
C: an n-variable arithmetic circuit over $\mathbb{F}_2$,
T: maximal parse subcircuit
μ: polynomial time perfect matching for the maximal parse subcircuits in C but T.

Output: An element $b \in \mathbb{F}_2^n$ satisfying C.

Theorem: MATCHED-CIRCUIT CNSS is in PPA

An instance of Matched-Circuit CNSS

Input: $N = (C, T, \mu)$ Remark: $C(x) = x_1 x_2$

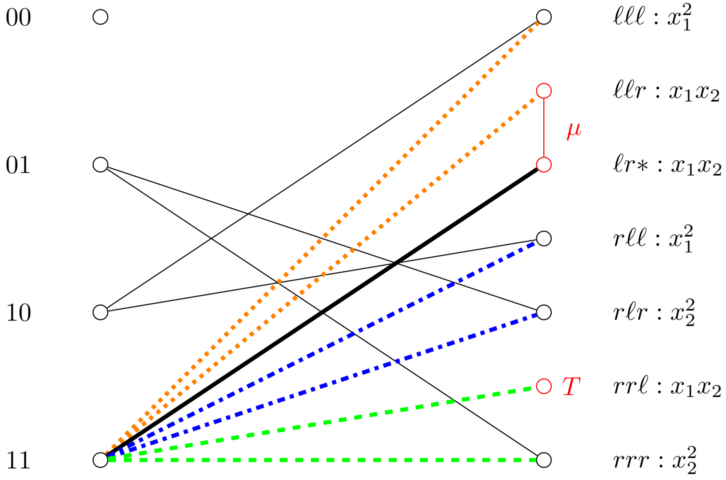


μ matches $\ell\ell r$ and $\ell r*$ unmatched $T = rr\ell$

PPA-easiness

Theorem: MATCHED-CIRCUIT CNSS is in PPA

Proof: We reduce MATCHED-CIRCUIT CNSS to LEAF.

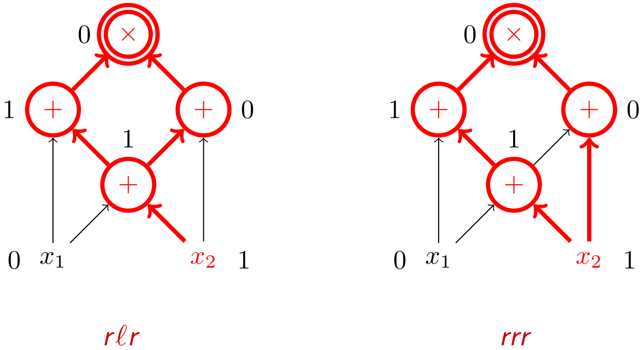


G_N resulting from the CIRCUIT-CNSS instance $N = (C, \mu, T)$

The pairing on the left hand side

Vertex 01 of even degree:

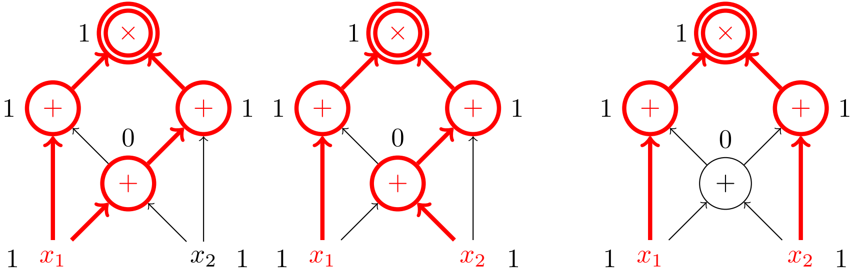
For all parse subcircuit S, $m_S(a) = 1$, $\exists$ sum gate g with $P_g(a) = 0$



The pairing on the left hand side

Vertex 11 of odd degree:

There exists a unique S, $m_S(a) = 1$, such that $P_g(a) = 1$ for all sum gate g



$\ell\ell\ell$ $\ell\ell r$ unique unmatched parse subcircuit

Thank you