Limitations of the Shpilka-Volkovich generator

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1. Polynomial Identity Testing
2. Shpilka-Volkovich (SV) Generator
3. Finding the Annihilating polynomial

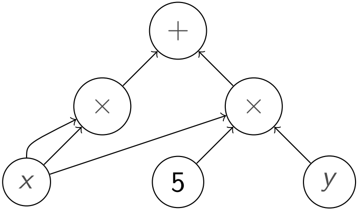
Section 1

Polynomial Identity Testing

Polynomial Identity Testing (PIT)

- *PIT:* Is a given input polynomial identically zero?

Input model: Arithmetic circuits



- A natural and succinct representation of a polynomial.

Blackbox test (a.k.a. Hitting set)

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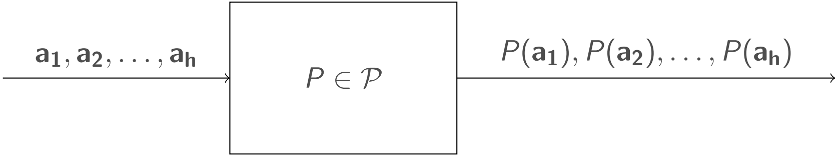

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- $\mathbf{a_i} = (a_{i1}, a_{i2}, \ldots, a_{in})$.
- Example: PIT for univariate polynomials of degree bounded by d.

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- $\mathbf{a_i} = (a_{i1}, a_{i2}, \ldots, a_{in})$.
- Example: PIT for univariate polynomials of degree bounded by d.
- For n-variate $\mathcal{P}$, a small-degree univariate substitution is enough.

Hitting set generator

- For a family $\mathcal{P}$ of n-variate, a polynomial map to k-variate polynomials $(f_1(\mathbf{y}), f_2(\mathbf{y}), \ldots, f_n(\mathbf{y}))$ is a hitting set generator if for all polynomials $P(x_1, x_2, \ldots, x_n) \neq 0 \in \mathcal{P}$, $P(f_1(\mathbf{y}), f_2(\mathbf{y}), \ldots, f_n(\mathbf{y})) \neq 0$.
- Final time complexity $= (\delta d + 1)^k$, where d is the degree of f_is and the polys in $\mathcal{P}$ are of degree δ.
- Poly when k is constant. Quasipoly when k is $\log n$.

Section 2

Shpilka-Volkovich (SV) Generator

Applications of the SV generator

- $s^{O(1)}$-size hitting set for Read-once formulas [Shpilka and Volkovich, 2009, Minahan and Volkovich, 2016].
- $s^{O(1)}$-size hitting set for Constant-read multilinear formulas [Anderson et al., 2015].
- $s^{O(\log\log s)}$-size hitting set for Commutative Read-once ABPs [Forbes et al., 2014].

Polynomials for Lagrange interpolation

- Building blocks of the SV generator.
- Choose $(a_1, a_2, \ldots, a_n)$ such that all a_is are unique.
- $L_r(y) := \prod_{j \neq r} \frac{(y - a_j)}{(a_r - a_j)}$.
- $L_r(b) = \begin{cases} 1 \text{ if } b = a_r, \\ 0 \text{ if } b \in \{a_1, a_2, \ldots, a_n\}, b \neq a_r. \end{cases}$

Shpilka-Volkovich map ($\mathsf{SV}_{n,k}$)[Shpilka and Volkovich, 2009]

- $\mathsf{SV}_{n,1}(y,z) : \mathbb{F}[\mathbf{x}] \longrightarrow \mathbb{F}[y,z]$, given by $\mathsf{SV}_{n,1}(y,z) : x_r \longmapsto zL_r(y)$.
- $SV_{n,1}$ is a bivariate map.
- $\mathsf{SV}_{n,k}(y_1,z_1,y_2,z_2,\ldots,y_k,z_k) : \mathbb{F}[\mathbf{x}] \longrightarrow \mathbb{F}[\mathbf{y},\mathbf{z}]$, given by $\mathsf{SV}_{n,k}(\mathbf{y},\mathbf{z}) : x_r \longmapsto \sum_{i=1}^{k} z_i L_r(y_i)$.
- $SV_{n,k}$ is a $2k$-variate map.

Some properties

- $\mathrm{SV}_{n,k}$ of each x_i is a linear form[1] in $\mathbf{z}$.
- $\mathrm{SV}_{n,k}$ is a hitting set generator for 2^k-sparse polynomials.
- $\mathrm{SV}_{n,k}$ is a hitting set generator for degree-k polynomials.

[1] constant part of the linear polynomial is 0

Question

- We want f such that $\mathsf{SV}_{n,k}(f) = 0$.
- What is the smallest degree polynomial that evaluates to 0 at $\mathsf{SV}_{n,k}$?
- Conjecture: There *exists* a degree $k+1$ multilinear polynomial on $n = 2k+1$ variables that maps to 0 on applying $\mathsf{SV}_{n,k}$.
- I.e. A multilinear, degree $k+1$ annihilating polynomial for $\mathsf{SV}_{n,k}$ exists.

Section 3

Finding the Annihilating polynomial

Finding a small annihilating polynomial - homogeneity

- Let
$$f(x_1, x_2, \ldots, x_n) = \sum_{S:|S|\leq k+1} \gamma_S \prod_{r\in S} x_r.$$
- Recall that $\mathrm{SV}_{n,k}(\mathbf{y}, \mathbf{z}) : x_r \longmapsto \sum_{i=1}^{k} z_i L_r(y_i)$.
- The polynomial $\mathrm{SV}_{n,k}(f)$ can be seen as a polynomial in $\mathbb{F}[\mathbf{y}][\mathbf{z}]$.
- After the map is applied, the $\mathbf{z}$-degree of a degree-d monomial is d.
- So, without loss of generality, f is homogeneous.

$$f(x_1, x_2, \ldots, x_n) = \sum_{S:|S|=k+1} \gamma_S \prod_{r\in S} x_r.$$

Coefficients of each monomials

- The coefficient of any such monomial after the map should be 0.
- This gives a set of linear constraints on γ_Ss.

… After some calculations

- After cleaning the conditions on the coefficients, our problem reduces to finding $(\alpha_R)_{R:|R|=k}$ such that the following linear constraints are satisfied:

$$\forall S \subseteq [n], |S| = k-1 : \sum_{R:|R|=k, S \subseteq R} \alpha_R = 0$$

and

$$\sum_{R:|R|=k, S \subseteq R} a_{R \setminus S} \cdot \alpha_R = 0.$$

- E.g. when $k = 1$, $n = 2k+1 = 3$. Then, we want to find $(\alpha_1, \alpha_2, \alpha_3)$ such that $\sum \alpha_i = 0$ and $\sum a_i \cdot \alpha_i = 0$.

- The $(\alpha_R)_R$s that satisfy the first set of constraints are nullvectors of the *inclusion matrix* M with the rows indexed by $\{S : |S| = k-1\}$ and the columns indexed by $\{R : |R| = k\}$ with

$$M_{S,R} = \begin{cases} 1 \text{ if } S \subseteq R, \\ 0 \text{ otherwise.} \end{cases}$$

- The $(\alpha_R)_R$s that satisfy the second set of constraints are null vectors of N, where

$$N_{S,R} = \begin{cases} a_{R \setminus S} \text{ if } S \subseteq R, \\ 0 \text{ otherwise.} \end{cases}$$

- When $k = 1$, $n = 2k + 1 = 3$,

$$M = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

and

$$N = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}.$$

- $N = D'MD$, where
  - D' is a $\binom{[n]}{k-1}$ diagonal matrix. $D'_{S,S} = \prod_{i \in S} 1/a_i = 1/a_S$.
  - D is a $\binom{[n]}{k}$ diagonal matrix. $D_{R,R} = \prod_{i \in R} a_i = a_R$.
- D' does not affect the nullvector of N.
- Hence, we need to show that

$$\mathcal{N}(M) \cap \mathcal{N}(MD) \neq \emptyset.$$

- $\mathcal{N}(M)$ has dimension $\binom{n}{k} - \binom{n}{k-1}$ and has been described by [Graham et al., 1980] and others.

[Graham et al., 1980]

- Some notation [Graham et al., 1980]:
  - View the nullvector as a multilinear homogeneous polynomial of degree k.
  - Take n variables $\{x_1, x_2, \ldots, x_n\}$. With a vector $(\alpha_R)_R$, associate $\sum_R \alpha_R x_R$.
  - Define

$$g(x_1, x_2, \ldots, x_n) = (x_1 - x_2)(x_3 - x_4) \cdots (x_{2k-1} - x_{2k})$$

.

- Lemma[Graham et al., 1980]

$$\mathcal{N}(M) = \text{span}\{g^\sigma | \sigma \in S_n\}$$

where, $h^\sigma(x_1, x_2, \ldots, x_n) = h(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$.

- Thus, $\mathcal{N}([1 \quad 1 \quad 1]) = \text{span}\{x_1 - x_2, x_1 - x_3, x_3 - x_2, \ldots\}$.

- $Mv = MDD^{-1}v$.
- Thus, $\mathcal{N}(MD) = \text{span}\{\psi(g^\sigma)|\sigma \in S_n\}$, where $\psi : x_i \longmapsto \frac{1}{a_i}x_i$.
- Let $b_i = \frac{1}{a_i}$.
- Thus, $\mathcal{N}([a_1 \quad a_2 \quad a_3]) = \text{span}\{b_1x_1 - b_2x_2, b_1x_1 - b_3x_3, b_3x_3 - b_2x_2, \ldots\}$.

- Conjecture: $\dim(\mathcal{N}(M) \cap \mathcal{N}(MD)) = 1$.
- When $k = 1, n = 3$, this common nullvector is

$$a_1 a_2(x_1 - x_2) + a_2 a_3(x_2 - x_3) + a_3 a_1(x_3 - x_1)$$

$$= -a_3(a_1 x_1 - a_2 x_2) - a_1(a_2 x_2 - a_3 x_3) - a_2(a_3 x_3 - a_1 x_1).$$

Thank you

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