Discovering the roots: Uniform closure results for algebraic classes under factoring

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Introduction

- For given input $f \in \mathbb{F}[x_1, \ldots, x_n]$, goal is to relate "complexity" of its factors and possibly output it.

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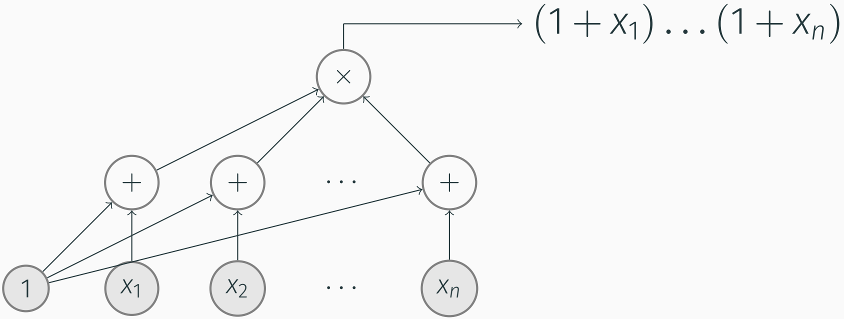
- We will be talking about different algebraic models of computation throughout. One of the most important is the "circuit" model.

Arithmetic Circuits



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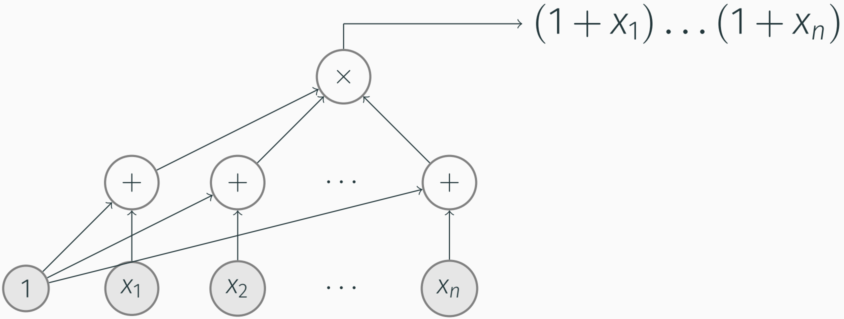

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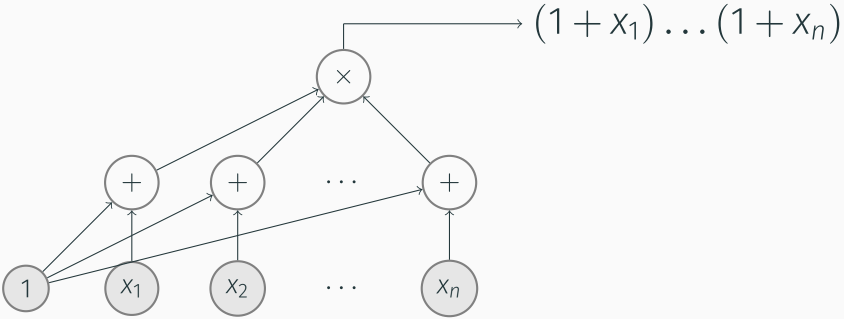

- size= # of nodes + # of edges = $5n + 2$
- # of monomials = 2^n

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  $$f = x^2y + x^3y^2 + xy + 4$$
  Here $\deg(f) = 5$
- $\mathrm{size}(f)$ denotes the minimum size of circuit computing f
- $f^{\leq d}$ denotes degree of f upto d i.e.
  $$f^{\leq d} = f \bmod \langle \bar{x} \rangle^{d+1}$$

Factorization in Circuit Models

Question: Given $f \in \mathbb{F}[\overline{x}]$ of $\text{size}(f) \leq s$, $\deg(f) = d$, what can we say about the size of its factors? 😫

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In other words, VP is *uniformly closed* under factoring!

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where ζ denotes 2^n-th root of unity.
- (LS'78) f_n has $O(n)$ size circuit but there are factors which has size $\geq \Omega(\frac{2^{n/2}}{\sqrt{n}})$.

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- Let $g = \prod_{i \in S} (x - \zeta^i)$ where $S \subset [2^n]$ with $|S| = n^{O(1)}$

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- Let $g = \prod_{i \in S} (x - \zeta^i)$ where $S \subset [2^n]$ with $|S| = n^{O(1)}$
- Trivially g has poly(n) size circuit!

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- (Kaltofen'87) If $f = g^e$, $\text{size}(f) = s$, $\deg(g) = d$; then $\text{size}(g) \leq \text{poly}(s, d)$. This is true over character 0 or field of large characteristic.

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- What can we say about factors of $f = g_1^{e_1} g_2^{e_2}$ where $\text{size}(f) = s$, $\deg(g_1), \deg(g_2) \leq d$? 😫

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- The degree of square-free part is polynomially bounded $\implies$ size "any" factor is!(and factor conjecture is true in this case!) 😄
- This subsumes both the results of Kaltofen

Factoring Reduces to Root Approximation

Finding linear factor

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- Can we do similar thing to find g? If yes, what is the notion of `approximation` ? What is the `starting point`? 😫

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- If $f'(\overline{x}, y_t)$ is invertible, then one can show that

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- If $f(\overline{0}, \mu) = 0$ and $f'(\overline{0}, \mu) \neq 0$. Then, one can find g by calculating $y_{\log d+1}$ where $\deg(g) = d$.

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- What about $f(\overline{x}, y) = \left(y^k + c_{k-1}(\overline{x})y^{k-1} + \ldots + c_0(\overline{x})\right) \cdot u$ where $k > 1$? 😫

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- So g is a root of $f(x+1,z) \in \mathbb{F}[[x]][z]$ as $f(x+1,g) = 0$
- Note that $z^2 - (x+1)^3 = (z - g^{\leq 3})(z + g^{\leq 3}) \bmod x^4$

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$\tau : x_i \mapsto x_i + \alpha_i y + \beta_i$, where $\alpha_i, \beta_i \in_r \mathbb{F}$, $\deg(\mathrm{rad}(f)) = d_0$,

$$f(\tau\overline{x}) = k \cdot \prod_{i \in [d_0]} (y - g_i)^{\gamma_i}$$

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- $f(x_1 + \alpha_1 y, \ldots, x_n + \alpha_n y)$ makes f monic in y
- For irreducible h, one can show that

$$h(\tau\overline{x}) = c \cdot \prod_{i=1}^{\deg(h)} (y - g_i)$$

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- If $\deg(h) = d_h \implies \deg(h(\tau\bar{x})) = d_h$

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- If $\deg(h) = d_h \implies \deg(h(\tau\bar{x})) = d_h$
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- Apply τ^{-1} on $h(\tau\bar{x})$ to get back $h(\bar{x})$.

Simultaneous Root Approximation (allRootsNI)

Are we done?

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- Are we done?
- If $f = (y - g)^e \cdot u$, to find g, we have to differentiate $e - 1$-times (wrt y). What is the size of $f^{(e-1)}$?

Derivative Computation

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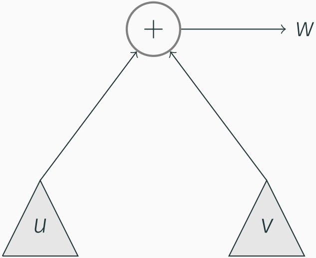
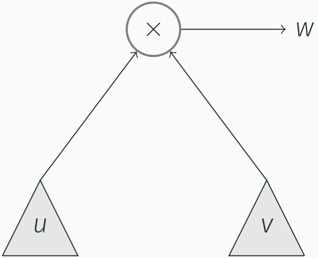
Proof Idea.

Compute inductively from bottom to top calculating upto k-th derivative i.e. at some node calculating u in the actual circuit, we keep track of $(u, u^{(1)}, \ldots, u^{(k)})$ instead! $\square$



$$w^{(i)} = u^{(i)} + v^{(i)}$$



$$w^{(i)} = \sum_{\mu=0}^{i} \binom{i}{\mu} u^{(i-\mu)} v^{(\mu)}$$

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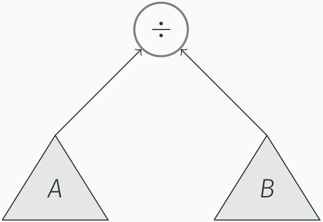

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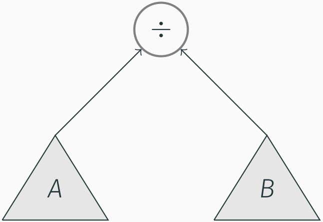

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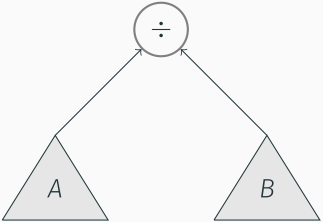

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- We don't know how to calculate this! 

Strassen's Division Elimination

- Can we find $\frac{A}{B} \bmod \langle \overline{x} \rangle^{d+1}$ if B is invertible?
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    4. there can be at most d_0 many factors f_i's!

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- $g_j^{\leq d_0}$ has poly(s, d_0)-size circuit $\implies$

$$f_i \equiv \prod (y - g_j^{\leq d_0}) \bmod \langle \overline{x} \rangle^{d_0+1}$$

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- Rearranging we have

$$\sum_{i \in [d_0]} \frac{e_i}{(y - \mu_i)^2} \cdot g_i^{=k} \equiv \frac{f'}{f} - \sum_{i \in [d_0]} \frac{e_i}{y - g_i^{\leq k-1}} \mod \langle \overline{x} \rangle^{k+1}$$

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- So the idea is solve each step without the mod and take the cumulative sum

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- One can show that it has $\text{poly}(s, d_0)$ size circuit with division

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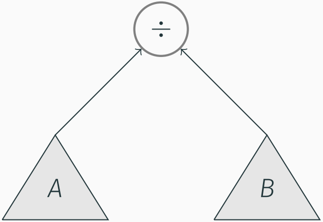
- In this case, we can show that after d_0 steps, the resulting division circuit has invertible denominator.



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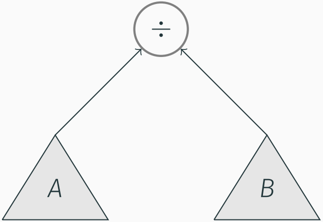

- This is because f evaluated at c_j's are invertible

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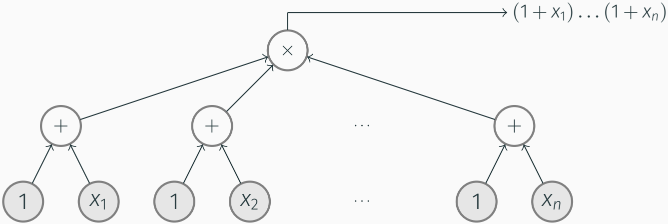
- Hence any irreducible factor (hence any factor) has $\text{poly}(s, d_0)$-size circuit 😄 😄

Some closure results

Arithmetic Formula



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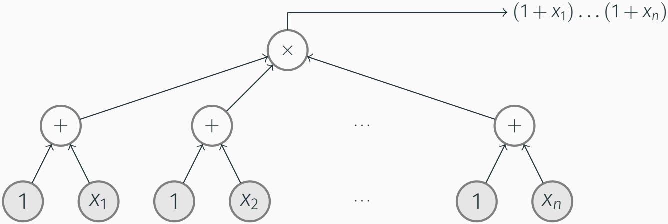

- Tree
- Leaves containing variables or constants

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Closure Results

Quasi-poly sized algebraic classes

$\{f_n\}_n \in \mathsf{VF}(n^{\log n})$ (resp. $\mathsf{VBP}(n^{\log n})$) such that n-variate f_n can be computed by an algebraic formula (resp. ABP) of size $n^{O(\log n)}$ and has degree poly(n).

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$\mathrm{VF}(n^{\log n})$ (resp. $\mathrm{VBP}(n^{\log n})$) is *closed* under factoring.
Moreover,there exists a `randomized` poly($n^{\log n}$)-time algorithm that: for a given $n^{O(\log n)}$ sized formula (resp. ABP) f of poly(n)-degree, outputs $n^{O(\log n)}$ sized formula (resp. ABP) of a nontrivial factor of f (if one exists).

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- differentiate $e-1$ times and use NI
- $g^{\leq d}$ will have size $n^{O(\log n)}$ formula
- Algorithm is non-trivial, uses idea by kaltofen

Closure of VNP

Definition of VNP

A family $\{f_n\}_n$ is in VNP if there exist polynomials $s(n), t(n)$ and a family $\{g_n\}_n$ in VP such that for every n,
$f_n(\bar{x}) = \sum_{w \in \{0,1\}^{t(n)}} g_n(\bar{x}, w_1, \ldots, w_{t(n)})$ where $\mathrm{size}(g_n) \leq s(n)$.

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THANK YOU! 😄